Overview of Transformations

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The term *geometric transformation* is typically used to describe a one-to-one function from a geometric space to itself. The types of geometric transformations that are a studied in the CaCCSS geometry standards are linear transformations on $\mathbb{R}^2$ (or $\mathbb{R}^3$), that is, transformations that map lines (or planes) onto lines (or planes).

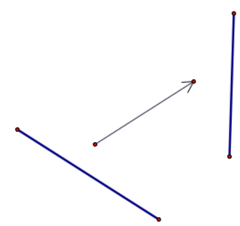
Definition 1. *A* linear transformation of the plane *is a transformation of the plane onto itself that maps lines onto lines.*

Proposition 2. *The set of all linear transformations is a group under the (associative) operation of composition. That is, if f and g are linear transformations, then*

a) The composition $f \circ g$ is a linear transformation.

b) The identity transformation is a linear transformation.

c) f has an inverse, and the inverse is a linear transformation.



The following proposition is used frequently in studying situations in which transformations play a part. An example is the trigonometric connection discussed[!h] above.

Proposition 3. *Properties of linear transformations*

a) Linear transformations preserve midpoints

b) Linear transformations transform parallel lines to parallel lines

Whenever we study functions, we are interested in finding out which points to not move under a given function. These points are rather special, and are called *fixed points*.

Definition 4. *A* fixed point *of a transformation is a point that is not changed under the transformation. That is, given a transformation f, a point x is a fixed point of f if and only if it satisfies $f(x) = x$.*

A special set of transformations consists of transformations that preserve (that is, do not change) the length of segments; these are discussed next.

Isometries

Definition 5. *An* isometry *is a (linear) transformation that preserves distance. That is, if $\overline{AB}$ is a segment in the plane, and f is an isometry, then the image of the segment, $f(\overline{AB}) = \overline{A'B'}$ has the same length as the original segment $\overline{AB}$.*

As a consequence of preserving distance, isometries also preserve other geometric properties.

Proposition 6. *Given an isometry f and an angle $\angle ABC$, the measure of the image $\angle A'B'C'$ is equal to the measure of the original angle $\angle ABC$; If a point P is on the segment $\overline{AB}$ then its image P' is on the segment $\overline{A'B'}$; and if X, Y, Z are collinear points (in any order), then their images are also collinear. That is, every isometry preserves* *a) Angles;*

b) Betweeness; and

c) Collinearity.

Since every isometry is a transformation, the set of all isometries are a subset of the group of transformations. Moreover, this subset has itself the structure of a group (and so is a subgroup of the group of transformations.) This fact is fundamental in being able to prove the Fundamental Theorem of Isometries.

Proposition 7. *The set of all isometries of the plane are a group under the (associative) operation of composition. That is,*

a) The composition of two isometries is an isometry.

b) The identity function is an isometry;

c) Given an isometry f, its inverse f^{-1} exists and is an isometry; and

Some questions now arise naturally:

1. What are some examples of isometries?

2. What relationships are there between different isometries?

3. Can we describe all isometries in some way?

4. Are there common "building blocks" for isometries?

The following definition gives us a partial answer to question #1:

Definition 8. *Let L be a line in the Euclidean plane. A* reflection *in L of the plane is a linear transformation f satisfying the following: if P is a point not on the line L, and $P = f(P)$, then L is the perpendicular bisector of the segment $\overline{PP'}$. If P is on L, then P'=P.* [1]

Note that there are also *central reflections* (reflections about a point), but we will not discuss them here.

It turns out that the answer to question #4 is a resounding "yes"; in fact all isometries can be constructed using reflections via function composition (see the Fundamental Theorem on Isometry below). It turns out that reflections are "generators" of the group of isometries. Since this is so, we will use this property of isometries in order to define the remaining isometries; defining them in this manner becomes particularly useful in understanding their properties.

Sometimes we wish to observe and use *direction* in regard to a figure. For that purpose, a geometric figure may be given *orientation*. That is, if we label points on the figure (for example the vertices of a polygon) and trace the figure, then we say:

[1] It remains to be proved that the function defined here is indeed a linear transformation.

1. The figure has *positive orientation* if the direction traced is counterclockwise, and *negative orientation* if the direction traced is clockwise.

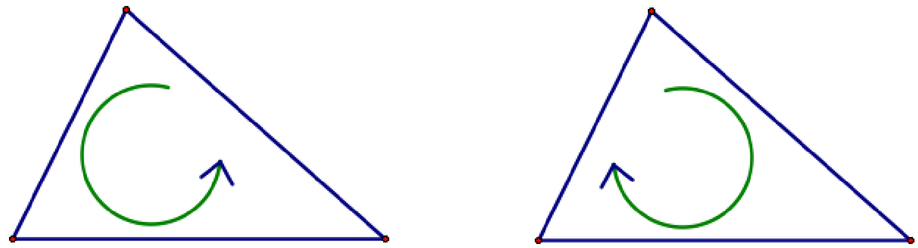


Figure 1: Positive Orientation (left) and Negative Orientation (right)

2. Some transformations will *preserve* orientation, and others (such as reflections) will *reverse* orientation.

3. A transformation is called *direct* if it preserves orientation, and *opposite* if it reverses orientation.

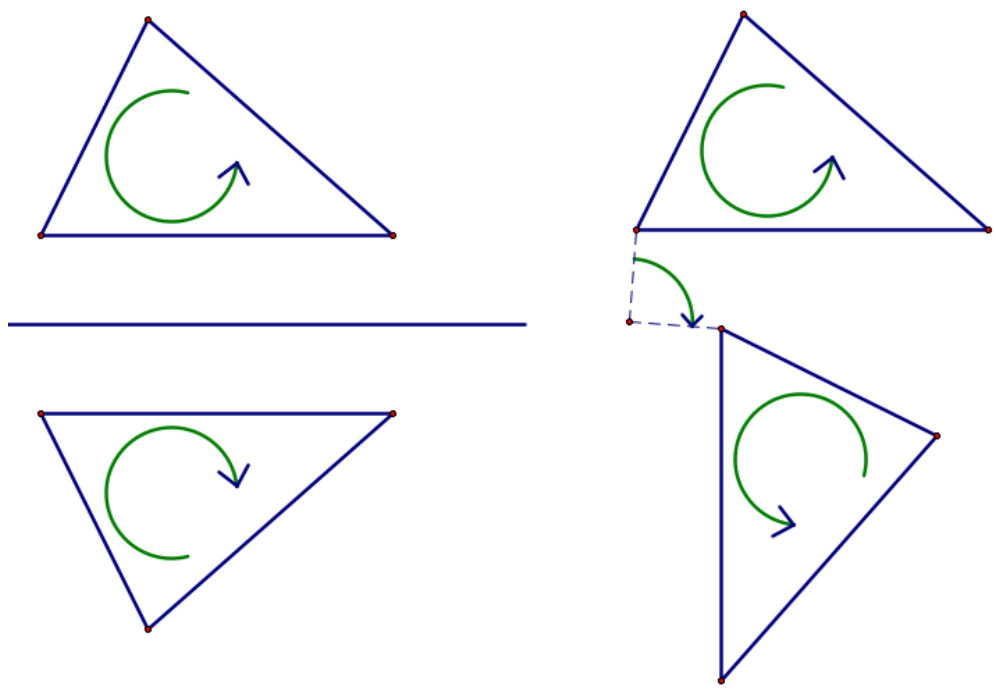


Figure 2: Reflection (left) reverses orientation and rotation (right) preserves orientation

Two types of isometries that preserve orientation are *translations* and *rotations*. Intuitively, a translation "slides" the plane in some direction on the plane. In a rotation we pick a point on the plane, and rotate the whole plane around it. We would like to define such motions of the plane as compositions of reflections, in order to understand them better, and indeed we do so next.

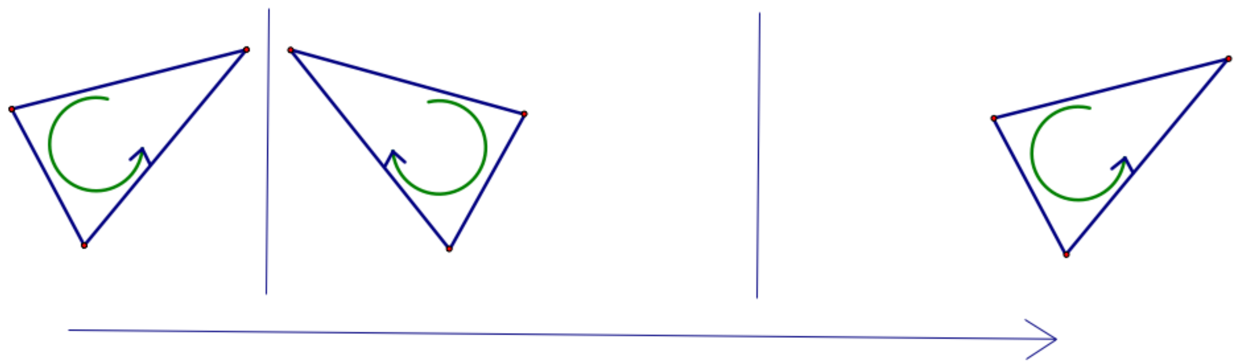


Figure 3: Translation as a product of two reflections

Definition 9. *1. A* translation *of the plane is a composition of two reflections s_l and s_m in parallel lines l and m.*

2. A rotation *of the plane is a composition of two reflection s_l and s_m in non-parallel lines l and m. The intersection of the lines l and m, point P, is called the* center of rotation.

3. A glide reflection *of the plane is a composition of a reflection and a translation.*

Proposition 10. *The product of two line reflections in parallel lines, s_l and s_m for parallel lines l and m, preserves* distance *and* slope.

The next theorem tells us that reflections are indeed the building blocks of isometries, since all isometries can be constructed as compositions of reflections - and in fact, of at most three reflections!

Theorem 11. Fundamental Theorem of Isometries *Every isometry in the plane is the product of at most* three *reflections. Additionally, an isometry is a product of exactly* two *reflections if it is direct and not the identity.*

Connection to Congruence It turns out that there is a fundamental and important relationship between isometry on the plane and congruence: Given two figures in the plane, they are congruent if and only if there is an isometry that maps one onto the other. Those familiar with Euclid's *Elements* might wish to consider the concept of "superposition" in relation to isometries and congruence.

Similarity Transformations

Until now we have discussed transformations that preserve distance. But what if we are given a linear transformation in which distance is not preserved? Is there anything we can say about the transformation? Indeed yes! In the following we list definitions and properties in which much is known, although distance is not preserved.

Definition 12. *A* similitude, *or* similarity transformation, *is a linear transformation such that for some non-zero constant k: $PQ = kPQ$ for all pairs of points P and Q and their images $P = s(P)$ and $Q = s(Q)$. The number k is called the dilation (or scale) factor for the similitude.*

What can happen for various values of k? If $k > 1$ distances grow, if $k = 0$ distances are preserved, and if $0 < k < 1$ distances get smaller. We sometimes say colloquially that a positive scale factor "stretches" figures and that a negative scale factor "shrinks" figures. Beware: this stretching or shrinking has to occur in all possible directions at once. Notice that the scale factor can also be negative; in this case, $|k|$ determines the size of the image, as described above. Its placement will be seen below.

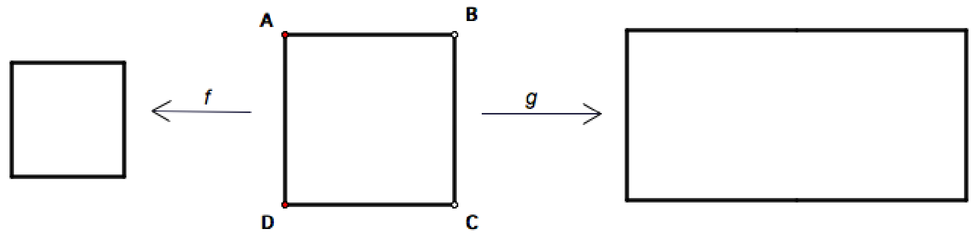


Figure 4: f is a dilation, but g is not

Note that a similarity transformation does not preserve distances except in the special case of $k = 1$, in which case it is in fact an isoemetry. We now ask: what properties of isometries are true for all similarity transformations? It turns out that quite a lot, in fact:

Proposition 13. *Similarity transformations preserve angle measure, betweeness, and collinearity.*

Having seen that all isometries are in fact a composition of (no more than three) reflections, we ask if similarity transformations can be constructed in some analogous way. Fortunately, the answer is yes! To see this, we first define a particularly important similarity transformation: a dilation. To get a sense of what a dilation is, imagine that you mark a special point in the plane (the center of dilation); call it O. Then stretch rays from O in every direction. For every point P in the plane, imagine it is a bead on the appropriate ray, and move the bead up or down the ray by a particular amount. Call the new point P' As long as your direction of movement is consistent, and the ratio $\dfrac{OP'}{OP}$ is constant, you have created a dilation.

An example of a dilation with positive scale factor $k = 2$ and a dilation with negative scale factor $k = -2$ is given below; to demonstrate the results of the dilation on the plane we have highlighted the triangle $\triangle ABC$ and its image in each case:

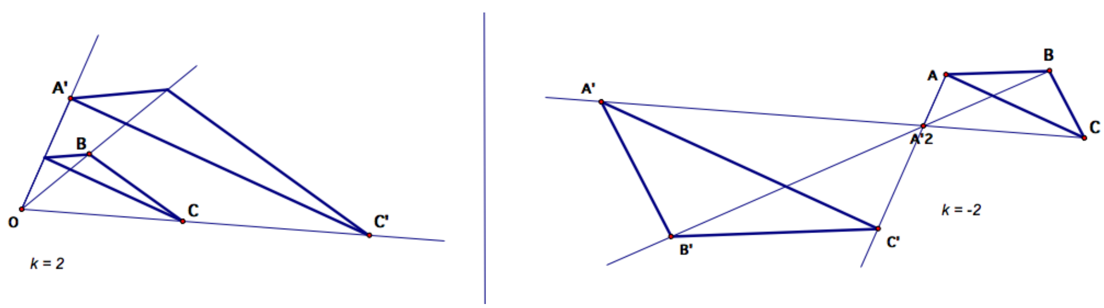


Figure 5: A dilation with positive scale factor (left) and negative scale factor (right)

Definition 14. *A* dilation *with center O and scale factor $k \neq 0$ is the unique similarity transformation that leaves O fixed and maps any other point P to the point P' on the line $\overleftrightarrow{OP}$ such that*

1. *If $k > 0$ then the point P is between O and P'*
2. *If $k < 0$ then the point O is between P and P'.*

and $\dfrac{OP}{OP'} = |k|$.

How are isometries, similarity mappings, and dilations connected? By its definition, a dilation is certainly a similarity mapping. What else can we say about the relationship between these?

Theorem 15. *Every similarity transformation is the composition of a dilation and an isometry, that is, it is the composition of a dilation and at most three reflections.*

Connection to Similarity Earlier, we saw that for every pair of congruent figures there is an isometry that maps one onto the other, and that every isometry maps figures onto congruent figures. A similar equivalence exists between similar figures and similarity transformations: For any two figures in the plane, they are similar if and only if there is a similarity transformation from one onto the other; conversely, every similarity transformation maps a figure onto a similar figure.

Common Misconceptions Regarding Transformations

1. A dilation (or any other similarity transformation) *must* change the size of segments. Another way to state the same misconception is that an isometry is not a similarity transformation. This is similar to the misconception that a square is not a rectangle.
2. Transformations involve moving figures in the plane, rather than moving all points of the plane
3. Given a 3-dimensional figure, scale only in one or two directions rather than all three dimensions
4. Orientation changes geometric properties of a figure